WEAKLY AND NON-WEAKLY BAND DOMINATED OPERATORS, AFTER ŠPAKULA, TIKUISIS AND ZHANG

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Contents

These are the notes for the NCG seminar, Spring 2019, of UH Manoa. This semester is devoted to the paper of Špakula, Tikuisis and Zhang on quasi-locality. These are the references:

- Relative commutant pictures of Roe algebras, Jan Špakula and Aaron Tikuisis [6],
- Quasi-locality and property A, Jan Špakula and Jiawen Zhang [7].

1. Approximation of band dominated operators

In the following, X denotes a discrete metric space (e.g. $\mathbb{Z}$) with bounded geometry. This last requirement means that, for each positive number r, the cardinality of the r-balls is uniformly bounded, i.e. the number $N_r = \sup_{x\in X} |B(x,r)|$ is finite. For $T \in B(\ell^2 X)$, define the *matrix coefficients* of T by

$$T_{xy} = \langle \delta_x, T\delta_y \rangle \quad \forall x, y \in X.$$

Think of T as a matrix (T_{xy}) indexed by X. The propagation of such a T will then be the (possibly infinite) number

$$prop(T) = \inf\{d(x,y) \mid T_{xy} \neq 0\}.$$

If the propagation of T is finite, we will say that T is *bounded* or has *finite propagation*. *Band dominated operators* are the norm limits of bounded operators. They form a C^*-algebra $C^*_u(X)$, called the *uniform Roe algebra* of X.

Questions:

(1) If T is band dominated, how can we approximate it by bounded operators?
(2) How can we recognize when T is band dominated?

The two next numbers will give partial answers to these two questions. Or at least try to explain why they are not trivial.

2. Approximation by bounded operators

For $T \in B(\ell^2 X)$ band dominated, define $T^{(n)}$ to be the operator with matrix coefficients

$$T^{(n)}_{xy} = \begin{cases} T_{xy} & \text{if } d(x,y) \leq n \\ 0 & \text{otherwise.} \end{cases}$$

We hope that T^n converges to T in norm as n goes to ∞.

As an example, take $X = \mathbb{Z}$ with its canonical metric (given by the absolute value). Each $f \in C(\mathbb{S}^1)$ gives rise to a multiplication operator $M_f \in B(L^2(\mathbb{S}^1))$, and by Fourier transform to a convolution operator $T_f \in B(\ell^2\mathbb{Z})$. It is the operator of norm $\|f\|_\infty$ with matrix coefficients $(T_f)_{xy}$ proportional to $\hat{f}(x-y)$.
In particular, if $f = \sum_{n=-N}^{N} \lambda_n z^n$ is a trigonometric polynomial, then T_f is bounded as $\hat{f}(n) = 0$ for $|n| > N$. This ensures that every T_f is band dominated, as every continuous functions is a uniform limit of trigonometric polynomials. For such operators, our naive guess

$$\text{“ } T_f^{(n)} \to_{\|\|} T_f \text{ ”}$$

is equivalent to

$$\text{“ } \sum_{n=-N}^{N} \hat{f}(n) z^n \to_{\|\|_\infty} f \text{ ”}$$

which is false. It is even worse: one can have $\|T_f\| = 1$ while $\|T_f^{(n)}\|$ goes to ∞ (Baire category argument, see [8] p. 167) and this implies (by uniform boundedness theorem) that $(T_f^{(n)})_n$ does not even converges to T_f in the strong operator topology.

3. Weakly band dominated operators

Definition 3.1. *An operator $T \in B(\ell^2 X)$ has (r, ε)-propagation if for every subsets $A, B \subset X$ such that $d(A,B) > r$,*

$$\|\chi_A T \chi_B\| < \varepsilon.$$

T is weakly band dominated if, for every $\varepsilon > 0$, there is $r > 0$ such that T has (r, ε)-propagation.

Note: Bounded implies weakly band dominated, therefore, weakly band dominated being a closed condition, band dominated implies weakly band dominated, as the intuition suggests.

Question: Does weakly bounded implies bounded?

This was claimed without proof for spaces with finite asymptotic dimension by J. Roe ca '97, and actually proved

- by Rabinovich-Roch-Silbermann in '00 for $X = \mathbb{Z}^n$ [4];
- by Špakula-Tikuisis in '16 for finite asymptotic dimension (and a bit more, finite decomposition complexity spaces for the curious reader) [6];
- by Špakula-Zhang in '18 for spaces with property A [7].

We have no counterexamples to this date (25 jan. 2019).

Theorem 3.2 (Folklore)**.** *The following are equivalent:*

(1) T is weakly band dominated;
(2) for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $f \in l^\infty(X)_1$ and $Lip(f) \leq \delta$ then $\|[T, f]\| < \varepsilon$.

Proof. Let us start with the reverse implication. Say $d(A,B) > r$, then there exists $f \in l^\infty(X)_1$ satisfying $0 \le f \le 1$, $f_{|A} = 1$, $f_{|B} = 0$ and $Lip(f) \le \frac{1}{r}$. Then $f\chi_A = \chi_A$ and $f\chi_B = 0$ so that

$$\chi_A T \chi_B = \chi_A [f, T] \chi_B$$

and $\|\chi_A T \chi_B\| \le \frac{1}{r}$.

Remark: the function

$$f(x) = \max\{0, 1 - \frac{d(x,A)}{r}\}$$

does the job. The Lipschitz constant is smaller than $\frac{1}{r}$ because of the easy inequality

$$|d(x,A) - d(y,A)| \le d(x,y) \quad \forall x, y \in X, A \subset X.$$

□

4. Characterizing membership in the Roe algebra

The main goal of this section is to prove the following result, after the work of Špakula and Tikuisis.

Theorem 4.1. *Consider the following properties of an operator $b \in B(\ell^2 X)$.*

(1) $\lim \|[b, f_n]\| = 0$ for every very lipschitz sequence $(f_n) \subset C_b(X)$;
(2) b is quasi local;
(3) $[b, g] \in \mathfrak{K}(\ell^2 X)$ for every Higson function $g \in C_h(X)$;
(4) $b \in C^*_u(X)$.

Then (4) $\implies$ (1) $\iff$ (2) $\iff$ (3)*. Moreover if X has finite asymptotic dimension, then* (4) *is equivalent to all of these.*

Some remarks are in order.

- These results grew out of a question of John Roe, who asked about the implication (2) $\implies$ (4) when X has *finite asymptotic dimension* (FAD).
- The theorem in [6] is better: (2) $\implies$ (4) when X has straight *finite decomposition complexity* (FDC), which is much weaker than FAD. For instance, $\mathbb{Z} \wr \mathbb{Z}$ has FDC but not FAD, while FAD always implies FDC.
- There is a follow up paper which shows (2) $\implies$ (4) when X has property A, an even weaker condition. This last result will be treated in a following number.

Let us understand the conditions better.

Very Lipschitz condition. Recall that a function f is Lipschitz if its Lipschitz modulus

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}$$

is finite. More precisely, a function f is L-Lipschitz if $Lip(f) \le L \iff |f(x) - f(y)| \le Ld(x,y), \quad \forall x \neq y$.

A sequence $(f_n) \subset l^\infty(X)$ is *very Lipschitz* if

- the sequence is uniformly bounded: $\exists C > 0$ such that $\|f_n\| \le C$;
- $\lim Lip(f_n) = 0$.

With this notation, the condition (1) is equivalent to

$$\forall \varepsilon > 0, \exists L > 0 \text{ s.t. if } \|f\| \le 1 \text{ and } Lip(f) \le L \text{ then } \|[b, f]\| < \varepsilon.$$

Indeed, one direction is obvious, and suppose there exists $\varepsilon > 0$ such that for every $L > 0$ there is a $f \in l^\infty(X)$ with $\|f\| \le 1$, $Lip(f) \le L$ and $\|[b, f]\| \ge \varepsilon$. Take $L = \frac{1}{n}$

to get a very Lipschitz sequence (f_n) with $\|[b,f]\| \geq \varepsilon > 0$, which contradicts (1).

Quasi-locality. Recall that $b \in B(\ell^2 X)$ is quasi-local iff $\forall \varepsilon > 0$, b has finite ε-propagation, iff $\forall \varepsilon > 0, \exists r > 0$ such that $\forall f, g \in l^\infty(X)$, if $\|f\|, \|g\| \leq 1$ and $d(supp(f), supp(g)) \geq r$ then $\|fbg\| < \varepsilon$.

Let us introduce the space

$$C_{L,\varepsilon} = \{a \in B(\ell^2 X) \ : \ \|[a,f]\| < \varepsilon \quad \forall f \in l^\infty(X)_1 \text{ s.t. } Lip(f) \leq L\}.$$

What was said above reduces to the fact that the algebra of quasi-local operators is exactly

$$\bigcap_{\varepsilon > 0} \bigcup_{L > 0} C_{L,\varepsilon}.$$

Higson functions. A function $g \in l^\infty(X)$ is said to be a Higson function, algebra denoted by $C_h(X)$ iff $\forall \varepsilon > 0, \forall r > 0$, there exists a finite subset $F \subset X$ such that if $x, y \notin F$ and $d(x,y) \leq r$, then $|g(x) - g(y)| \leq \varepsilon$.

4.1. **Roe's question on conditions (2) and (4).** (4) $\implies$ (2) is not difficult. In short, quasi-locality is a closed condition, which is obviously satisfied by finite propagation bounded operators.

Closed condition If $\forall \delta > 0$, there is a quasi-local operator b' such that $\|b - b'\| < \delta$, then b is quasi-local.

Finite propagation operators are quasi-local If $\xi \in \ell^2(X)$ is finitely supported and $prop(b) \leq r$, then $supp(b\xi) \subset N_r(supp(\xi))$, and so if $d(supp(f), supp(g)) > r$, then $gbf = 0$.

(4) $\implies$ (1) is again not too hard.
Condition (1) is closed and is satisfied by finite propagation operators. This follows from elementary estimates and a calculation of the kernel of the commutator.

Lemma 4.2. *If $b \in B(\ell^2 X)$ such that $|b(x,y)| \leq C$ and $prop(b) \leq r$, then $\|b\| \leq CN_r$*

Lemma 4.3. *Let b as above and $f \in l^\infty(X)$. The kernel of $[b,f]$ is*

$$[b,f](x,y) = b(x,y)(f(x) - f(y))).$$

Now (4) $\implies$ (1) follows: if $prop(b) \leq r$, then

$$prop([b,f]) \leq r \text{ and } |[b,f](x,y)| \leq CLip(f)r$$

so that also $\|[b,f]\| \leq CrN_r Lip(f)$. As for the lemmas, the first point reduces to:

$$\begin{aligned} |b\xi(x)| &\leq \sum_{y \in B_r(x)} |b(x,y)| \ |\xi(y)| \\ &\leq CN_r^{\frac{1}{2}} \|\xi_{|B_r(x)}\|_2 \quad \text{by CBC.} \\ \implies \|b\xi\|_2^2 &= \sum_x |b\xi(x)|^2 \\ &\leq \sum_x C^2 N_r \|\xi_{|B_r(x)}\|_2^2 \\ &\leq \sum_x \sum_{y \in B_r(y)} C^2 N_r |\xi(y)|^2 \\ &\leq C^2 N_r^2 \|\xi\|_2^2 \end{aligned}$$

The second point is a direct calculation.

(1) $\implies$ (2). The key point is the following.

Lemma 4.4. *If $A, B \subset X$ such that $d(A,B) \geq r$, then there exists a function $\phi : X \to [0,1]$ such that*

- $\phi = 1$ *on* A,
- $\phi = 0$ *on* B,
- $Lip(\phi) \leq \frac{1}{r}$.

Proof. Let us show that it gives the claimed implication. Let $\varepsilon > 0$, condition (1) gives a constant L. Put $r > L^{-1}$. If then $f, g \in l^\infty(X)_1$ such that $d(supp(f), supp(g)) \geq r$ we have $f\phi = f$ and $g\phi = 0$ so that

$$\|fbg\| = \|f[\phi, b]g\| \leq \|[\phi, b]\| < \varepsilon.$$

As for the lemma, just take

$$\phi(x) = \max\{0, 1 - \frac{d(x,A)}{r}\}.$$

□

(2) $\implies$ (1). The key here idea is: if f has a small Lipschitz constant, then it varies slowly so that its level sets are well separated.

Let $f \in l^\infty(X)$ such that $0 \leq f \leq 1$ and $Lip(f) \leq L$, and put

$$A_i = \{x \mid \frac{i-1}{N} < f(x) \leq \frac{i}{N}\} \quad i = 2, N$$
$$A_1 = \{x \mid 0 \leq f(x) \leq \frac{1}{N}\}$$

Then $f \sim \sum_{i=1}^N \frac{i}{N} A_i := g$ (actually $\|f - g\| \leq \frac{1}{N}$) and also

$$d(A_i, A_j) \geq \frac{1}{NL} \quad \text{if } |i-j| \geq 2.$$

Also the A_i's are disjoint and cover X. we will now estimate $\|[b,g]\|$. Let $\varepsilon > 0$,

$$\begin{aligned}
\|[g,b]\| &= \|[\sum_i \frac{i}{N} A_i, b]\| \\
&= \| \left(\sum_i \frac{i}{N} A_i\right) \ b \ \left(\sum_i A_i\right) - \left(\sum_i A_i\right) \ b \ \left(\sum_i \frac{i}{N} A_i\right) \| \\
&= \| \sum_{i,j} (\frac{i}{N} - \frac{j}{N}) A_i b A_j \| \\
&\leq \| \sum_{|i-j|=1} \frac{1}{N} A_i b A_j \| + \| \sum_{|i-j|\geq 2} (\frac{i}{N} - \frac{j}{N}) A_i b A_j \|
\end{aligned}$$

Let us label the summands of this last line by I and II. By quasi-locality of b, we get a $r = r(\varepsilon) > 0$, then for any choice of N, put $L = L(N, \varepsilon)$ such that $L < (rN)^{-1}$. Any such f with $Lip(f) \leq L$ satisfies $d(A_i, A_j) > r$ so that $\|A_i f A_j\| < \varepsilon$ for each term in the second summand, so that

$$(II) \leq N^2 \varepsilon.$$

For (I), the pairs (i,j) can be split up into 4 classes: (i odd, $j = i+1$), (i even, $j = i+1$) and the two symmetric cases. For each of these families, the sum is a block sum with orthogonal domain and range, hence the norm of the sum is less than the sup of the norm of the terms, so that

$$(I) \leq \frac{4}{N}.$$

Let us wrap all of this up: if ε is given, choose N such that $\frac{4}{N} < \varepsilon$, choose $L = L(N, \frac{\varepsilon}{N^2})$. This gives:

$$\begin{aligned}\|[b,g]\| &\leq (I) + (II) \\ &\leq \frac{4}{N} + \frac{\varepsilon}{N^2} N^2 \\ &\leq 2\varepsilon.\end{aligned}$$

5. Heart of the paper

Let us turn the attention to the most important result of the paper:

$$\text{If } X \text{ has FAD, then } (1) \implies (4).$$

Theorem 5.1. *Let X be a bounded geometry uniformly discrete metric space. If X has finite asymptotic dimension, then*

$$\forall \varepsilon > 0, \exists L > 0 \ s.t. \ a \in Commut(L, \varepsilon) \implies a \in C^*_u(X)$$

and

$$a \in Commut(L, \varepsilon) \iff \|[a,f]\| < \varepsilon \quad \forall f \in l^\infty(X)_1, Lip(f) < L.$$

Review of asymptotic dimension. Recall that X has *asymptotic dimension* less than d if for every $r > 0$, there exists a bounded cover X which is (d,r)-separated. The typical example is the group $\mathbb{Z}$ with the metric induced by the absolute value, which has asymptotic dimension bounded by 1. As an exercise, prove that $asdim(\mathbb{Z}^n) \leq n$.

In the context of $asdim \leq 1$, conditional expectations into block subalgebras are very natural. Consider subsets $\{U_i\}$ of X which are pairwise disjoint and u_i the correponding multiplication operators. Define

$$\theta(a) = \sum_i u_i a u_i \quad \forall a \in B(\ell^2 X).$$

- $\theta(a)$ is SOT convergent;
- θ is lower continuous;

  Both of these follow essentially from

  $$\begin{aligned}\|\theta(a)\xi\|^2 &= \sum_i \|u_i a u_i \xi\|^2 \quad \text{by orthogonality of the support,} \\ &\leq \|a\| \ \sum_i \|u_i \xi\|^2 \\ &\leq \|a\| \ \|\xi\|^2\end{aligned}$$

  Take the directed systems of all the sums over finite subsets of I, in which case the sum is finite.
- θ is a conditional expectation. (Meaning it is CP, $\theta(xay) = x\theta(a)y$ when x, y are block diagonals wrt $\bigoplus_i \ell^2 U_i$, and $\theta(a)$ is block diagonal.)

Write $u = \sum_i u_i$.

Fact: if $prop(a) \leq r$ and $\mathcal{U}$ is $2r$-separated, then $uau = \theta(a)$.

Proof. If $\xi \in \ell^2 X$, $supp(u_i\xi) \subset U_i$, so that $supp(au_i\xi) \subset N_r(U_i)$ which is disjoint from U_j, $j \neq i$. Hence the cross terms $u_j a u_i \xi$ vanish. We get for finite sums

$$\sum_{i,j\in F} u_i a u_j \xi = \sum_{i\in F} u_i a u_i \xi,$$

and the result follows by continuity. $\square$

Consequence: If $b \in C_u^*(X)$, for every $\varepsilon > 0$, there exists $r > 0$ such that if $\mathcal{U}$ is r-separated, then

$$\|ubu - \theta(b)\| < \varepsilon.$$

This renders the next proposition natural.

Proposition 5.2. *(Cor 4.3) If $a \in Commut(L, \varepsilon)$, with the notations above, if $\mathcal{U}$ is $\frac{2}{L}$-separated, then*

$$\|uau - \theta(a)\| < \varepsilon.$$

Remark: if the theorem is true, then the above discussion shows that the result above must be true.

Proof. (of the theorem, assuming the above proposition) If $asdim(X) \leq 1$, fix a big $r > 0$: we get a bounded cover $\mathcal{Y}$ which is $(1, r)$-separated, meaning

$$\mathcal{Y} = \mathcal{U} \cup \mathcal{V}$$

with $\mathcal{U}$ and $\mathcal{V}$ r-separated families. Then, if $a \in B(\ell^2 X)$,

$$a = uau + uav + vau + vav.$$

we want to show that if $a \in Commut(L, \varepsilon)$ and $r > 4L^{-1}$, then each term on the right is near a finite propagation operator.

Claim: this is true for uau and vav.

This follows from the proposition: $\mathcal{U}$ is $r > 4L^{-1} > 2L^{-1}$-separated so that

$$\|uau - \theta(a)\| \leq \varepsilon$$

and $\theta(a)$ is block diagonal w.r.t. a bounded family, it is thus of finite propagation (less than $\sup_{\mathcal{U}} diam(U)$).

Claim: this is true for uav and vau.

Let $\mathcal{U}' = N_{L^{-1}}(\mathcal{U})$, same for $\mathcal{V}'$. Both are at least $2L^{-1}$-separated. We thus obtain $f = \sum f_i$ with f_i $[0,1]$-valued, with value 1 on U_i, 0 on $N_{L^{-1}}(U_i)^c$ and $Lip(f_i) \leq L$. Similarly for $\mathcal{V}$, we get $g = \sum_j g_j$. Then $uf = u$ and $vg = v$. Put

$$W_{ij} = N_{L^{-1}}(U_i) \cap N_{L^{-1}}(V_j) \quad , W = \coprod_{i,j} W_{ij}$$

which is at least $2L^{-1}$-separated. Similarly, build w and w_{ij}. we then calculate:

$$\begin{aligned} uav &= ufagv \\ &= ugafv + u[f,a]gv + u[a,g]fv \\ \text{So } \|uav - ugafv\| &\leq 2\varepsilon \end{aligned}$$

but now, $ugafv \in Commut(L, \varepsilon)$, so that the proposition applies using the $\{W_{ij}\}$ which are $2L^{-1}$-separated:

$$\|wugafvw - \theta_w(ugafv)\| \leq \varepsilon.$$

But $wugafvw = ugafv$ since $wugug$ and $fvw = fv$ ($supp(fv) \subset W$ and $supp(ug) \subset W$). And $\theta_w(ugafv)$ is block diagonal with finite propagation. $\square$

It remains to prove the proposition.

Block diagonal symmetries.

Lemma 5.3. *If $a \in C_{L,\varepsilon}$ and $\mathcal{U}$ is a $\frac{2}{L}$-separated family of X then*

$$\|[uau, \overline{u}]\| < \varepsilon$$

where $u = \sum u_i$ is our usual notation for the characteristic function of $\cup U_i$, and $\overline{u}$ is a block diagonal symmetry, i.e. an operator of the type $\sum \epsilon_i u_i$, $\epsilon_i \in \{-1, 1\}$.

Proof. Extend each u_i to a $[0,1]$-valued L-Lipschitz function, which is 1 on U_i and zero outside of $N_{L^{-1}}(U_i)$. Then the f_i have disjoint support so that

$$\overline{f} = \sum_i \epsilon_i f_i$$

satisfies $Lip(\overline{f}) \leq L$, $\|f\| \leq 1$ and $\overline{f}u = \overline{u}$. But

$$[uau, \overline{u}] = u[a, \overline{f}]u$$

which has norm smaller than ε. $\square$

The block diagonal symmetries form a topological group (with the SOT topology), isomorphic to $\prod_{\mathcal{U}} \mathbb{Z}_2$ endowed with the product topology. It is thus a totally disconnected compact group, and has a unique Haar probability measure $d\overline{u}$.

Lemma 5.4. *Let $Z \subset X$ and $b \in B(\ell^2 Z)$. If*

$$\|[b, \overline{u}]\| < \varepsilon \quad \forall \overline{u} \ \text{ block diagonal symmetry}$$

then $\|b - E(b)\| < \varepsilon$, where $E : B(\ell^2 Z) \to \bigoplus l^\infty(U_i)$ is the canonical expectation onto the block diagonal. Furthermore,

$$E(b) = \int_G \overline{u} b \overline{u} \ d\overline{u}.$$

Remark: the example of the two point space is helpful to understand what is happening. Let say

$$\overline{u} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

then a simple calculation shows

$$\begin{aligned} \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} &= \frac{1}{4} \sum_{\overline{u} \in G} \overline{u} b \overline{u} \\ &= \frac{1}{4} \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} x & -y \\ -z & w \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \begin{pmatrix} x & -y \\ -z & w \end{pmatrix} \right). \end{aligned}$$

The estimate is easy:

$$\|E(b) - b\| \leq \frac{1}{4} \sum \|b\overline{u} - \overline{u}b\| < \varepsilon.$$

Proof. First check that on the group G, the $*$-SOT, SOT and pointwise convergence coincide. Since the norms are all smaller than 1, we can consider finitely supported vectors, or even basis vectors.

Next the integral is understood in the weak sense, meaning that the assertion is

$$\langle E(b)\xi, \eta \rangle = \int_G \langle \ \overline{u} b \overline{u} \ \xi, \eta \rangle \ d\overline{u} \quad \forall \xi, \eta \in \ell^2 X.$$

Ignoring existence, check the following matrix coefficients

$$\langle E(b)\delta_i, \delta_j\rangle = \begin{cases} b_{ii} & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

and

$$\int_G \langle \overline{u} b \overline{u}\ \delta_i, \delta_j\rangle\ d\overline{u} = \int_G \langle b\overline{u}\ \delta_i, \overline{u}\ \delta_j\rangle d\overline{u}$$
$$= (\int_G \epsilon_i\epsilon_j\ d\overline{u})\ b_{ij}.$$

But

$$\int_G \epsilon_i\epsilon_j\ d\overline{u} = \mathbb{P}(\epsilon_i = \epsilon_j) - \mathbb{P}(\epsilon_i \neq \epsilon_j)$$

which is $\frac{1}{2} - \frac{1}{2} = 0$ if $i \neq j$, 1 otherwise. □

Finally let's put all the lemmas together to get the proposition.

Proof. Let $\mathcal{U}$ be a $\frac{2}{L}$-separated family and $a \in C_{L,\varepsilon}$.
The first lemma gives

$$\|[uau\ , \overline{u}]\| < \varepsilon \quad \forall \overline{u} \in G,$$

and now $uau \in B(\ell^2 Z)$ for $Z = \cup U_i$, so by the second lemma,

$$\|uau - E(uau)\| < \varepsilon.$$

The canonical expectation $E(uau)$ is $\theta_{\mathcal{U}}(a)$, and this concludes the proof. □

6. Property (A)

The last part of the section is devoted to prove the assertion (quasi-locality implies locality) when X has property (A).

Property (A) and its friends. *Motivation:* let G be a countable discrete group with a bounded geometry left-invariant metric d. For each $A \subset X$ and every $r > 0$, define the r-corona of A to be the set

$$\partial_r A = \{x \in X \mid 0 < d(x, A) \leq R\}.$$

Here is a possible definition of amenability.

Definition 6.1. *The group G is amenable if for all $r, \varepsilon > 0$, there exists a finite subset $A \subset X$ satisfying*

$$|\partial_r A| < \varepsilon |A|.$$

Remark: we don't suppose the group to be finitely generated. For instance $G = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ with the metric $l(n) = \sum_i i|n_i|$ is not finitely generated, yet is of bounded geometry and amenable. If G is finitely generated, one does not need to quantify over r in the definition and can use ∂A instead.

This definition of amenability makes perfect sense for any bounded geometry metric space. However, it is a bit silly, since for any bounded geometry space X, the space $X \cup \mathbb{N}$ is amenable. Indeed, given $r > 0$, take $A_r = [r, r+N] \subset \mathbb{N}$. Then $\frac{|\partial_r A|}{|A|} = \frac{2r}{N+1}$ is very small for N large. This definition of amenabilty is thus local (" something nice happens somewhere") when we actually want to say something about the global structure of X.

Definition 6.2. *The metric space X is uniformly locally amenable, abreviated $(ULA)_\mu$ after on, if for all $r, \varepsilon > 0$, there exists $s > 0$ such that for all probability measure $\mu \in Prob(X)$, there is a finite subset $A \subset X$ satisfying*

$$diam(A) \leq s \quad \textit{and} \quad \mu(\partial_r A) < \varepsilon\mu(A).$$

Remarks:

- The strict inequality is important, otherwise take $A = \emptyset$.
- The condition would be vacuous without the condition $diam(A) \leq s$, with s uniform on all probability measures. Otherwise just take the uniform probability on A for all A: the measure of the r-corona is 0.
- $(ULA)_\mu$ is equivalent to property (A), see [1].
- If G is a group, then if G is amenable, G is $(ULA)_\mu$. The proof is left as an exercise for the reader.

More definitions.

Definition 6.3 ([3])**.** *The metric space X is exact if for all $r, \varepsilon > 0$, there exists $s > 0$ and a partition of unity $\{\phi_i\}_i$ on X such that*

- *if $d(x, y) < r$, then*

$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \varepsilon,$$

- *$diam(supp(\phi_i)) \leq s$ for every $i \in I$.*

Definition 6.4 ([2])**.** *The metric space X has the metric sparsification property, abreviated MSP after on, if for all $r, \varepsilon > 0$, there exists $s > 0$ such that for all $\mu \in Prob(X)$, there exists $\Omega \subset X$ such that*

- *$\mu(\Omega) \geq 1 - \varepsilon$,*
- *Ω is a r-disjoint union of s-bounded sets.*

Theorem 6.5 (by everyone above)**.** *Exact* $\Longrightarrow_{(1)}$ $(ULA)_\mu$ $\Longrightarrow_{(2)}$ *MSP* $\Longrightarrow_{(3)}$ *Exact.*

The implication (3) is harder, see Sako [5]. The proof is C^*-algebraic: can we find a direct proof?

Proof. (1) Given $r, \varepsilon > 0$, let $\mu \in Prob(X)$, and $\{\phi_i\}$ be as in the definition with

$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \frac{\varepsilon}{N_r}.$$

Hence for each fixed x,

$$\sum_{y: d(x,y) \leq r} \sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \varepsilon = \varepsilon \sum_i \phi_i(x).$$

As μ is a probability measure,

$$\sum_x \mu(x) \sum_{y: d(x,y) \leq r} \sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \varepsilon \sum_x \mu(x) \sum_i \phi_i(x).$$

hence there exists an index i_0 such that

$$\sum_x \mu(x) \sum_{y: d(x,y) \leq r} |\phi(x) - \phi(y)| < \varepsilon \sum_x \mu(x) \phi(x).$$

with $\phi = \phi_{i_0}$. now write $\phi = \sum a_i \chi_{F_i}$ where $a_i > 0$ and $F_{i+1} \subset F_i$. All the F_i's are in $supp(\phi)$ so their diameter is bounded above by s.

$$\sum_x \mu(x) \sum_{y: d(x,y) \leq r} |\sum_k a_k(\chi_{F_k}(x) - \chi_{F_k}(y))| < \varepsilon \sum_x \mu(x) \sum_k a_k \chi_{F_k}(x)$$

$$\sum_x \mu(x) \sum_{y: d(x,y) \leq r} \sum_k a_k |\chi_{F_k}(x) - \chi_{F_k}(y)| < \varepsilon \sum_x \mu(x) \sum_k a_k \chi_{F_k}(x).$$

Hence

$$\sum_x \mu(x) \sum_{y:d(x,y)\le r} |\chi_{F_k}(x) - \chi_{F_k}(y)| < \varepsilon \sum_x \mu(x)\chi_{F_k}(x) = \varepsilon\mu(F_k)$$

for some $k = k_0$, and for $x \in \partial_r F_{k_0}$,

$$\sum_{y:d(x,y)\le r} |\chi_{F_k}(x) - \chi_{F_k}(y)| \le 1 \le \sum_{x\in\partial_r F_{k_0}} \mu(x) = \mu(\partial_r F_{k_0}).$$

Set $A = F_{k_0}$, then

$$\mu(\partial_r F_{k_0}) < \varepsilon\mu(A).$$

$\square$

Quasi-locality and property (A). The main goal of this section is to provide a proof of (1) $\implies$ (4) in the case where X has property (A). Let us fix some notations.

For (X, d) a metric space, a partition of unity will be given by a pair $(\phi, \mathcal{U})$ where $\mathcal{U}$ is a cover of X and ϕ is a map

$$\phi : X \to \ell^2(\mathcal{U})_{1,+} \quad ,$$

such that $x \mapsto \phi_U(x)$ is supported in U for every $U \in \mathcal{U}$. (The notation $\ell^2(\mathcal{U})_{1,+}$ means positive elements of norm 1.) If $\mathcal{U} = \{U_i\}_{i\in I}$, we will identify $\ell^2(\mathcal{U})$ with $\ell^2(I)$.

The characterization of property (A) which we use is the following, obtained by Dadarlat and Guentner in [3].

Theorem 6.6. *A metric space X is called exact if, for every $r, \varepsilon > 0$, there exists a partition of unity $\phi : X \to \ell^2(\mathcal{U})$ such that $\mathcal{U}$ is uniformly bounded with finite multiplicity and*

$$d(x, y) \leq r \implies \|\phi(x) - \phi(y)\|_2 \leq \varepsilon.$$

If X is discrete and of bounded geometry, exactness and property (A) are equivalent.

We will also need to know that property (A) implies the metric sparsification property, which was proven in the last section.

The key idea of the proof relies on an approximation property of quasi-local operators: their norm can be approximated by finitely supported vectors. This means that if $b \in \bigcap_\varepsilon \bigcup_L C_{L,\varepsilon}$,

$$\|b\| = \sup_{\|v\|=1\ ,\ diam(supp(v))<\infty} \|bv\|.$$

This relies on the following lemma.

Lemma 6.7 ([7], lemma 5.2)**.** *For every M, L, ε, there exists $s > 0$ such that, for every $b \in C_{L,\varepsilon}$ with $\|b\| \leq M$, there exists $v \in \ell^2(X)$ satisfying $\|v\| = 1$, $diam(supp(v)) < s$ and*

$$\|bv\| \geq \|b\| - \varepsilon.$$

Proof. (of the result, using the lemma) Let X discrete with bounded geometry and property (A), and say $b \in B(\ell^2 X)$ is quasi-local and fix $\varepsilon > 0$. Then there is $L > 0$ such that $b \in C_{L,\varepsilon}$ and, by the lemma, a $s > 0$ such that $\|T\|$ can be approximated up to ε by s-supported vectors for every $T \in C_{2\varepsilon,L}$ with $\|T\| \leq M$.

Choose a partition of unity ϕ with uniformly bounded support and

$$d(x, y) \leq s + \frac{1}{L} \implies \|\phi(x) - \phi(y)\| \leq \varepsilon.$$

Let us show that the norm of

$$a = b - \sum_i \phi_i b \phi_i = \sum_i \phi_i [\phi_i, b]$$

is small enough.

The following computation shows that $a \in C_{2\varepsilon,L}$:

$$\begin{aligned} \|[a, f]\| &= \|[\sum_i \phi_i[\phi_i, b], f]\| \\ &\leq \| \sum_i \phi_i [b, f] \phi_i\| + \|[b, f]\| \\ &\leq 2\varepsilon \end{aligned}$$

where we used $\|\sum_i \phi_i[b,f]\phi_i - \phi[b,f]\phi\| < \varepsilon$. This follows from the fact that, if e_j are positive contractions with $\frac{2}{L}$-separated support, and $T \in C_{\varepsilon,L}$, then $\|eTe - \sum_i e_i T e_i\| < \varepsilon$. This is not a trivial statement, and was proven in the last section (Cor 5.3 of [7]).

Of course, $\|a\| \leq 2M$, so we can apply the statement of the first paragraph to a: there exists a unit vector $v \in \ell^2 X$ with support F satisfying $diam(F) < s$ and $\|av\| \geq \|a\| - \varepsilon$, and

$$
\begin{aligned}
|\sum_i \phi_i(x)(\phi_i(x) - \phi_i(y))b_{xy}| &\leq M(\sum_i \phi_i^2(x))^{\frac{1}{2}} \ (\sum_i |\phi_i(x) - \phi_i(y)|^2)^{\frac{1}{2}} \\
&\leq M\|\phi(x) - \phi(y)\|_2
\end{aligned}
$$

so that if $x \in N_{L^{-1}}(F)$, $\|\phi(x) - \phi(y)\|_2 \leq \varepsilon$, and

$$
\begin{aligned}
|av|_x &= |\sum_{i,y\in F} \phi_i(x)(\phi_i(x) - \phi_i(y))b_{xy}v_y| \\
&\leq \sum_{y\in F} |\sum_i \phi_i(x)(\phi_i(x) - \phi_i(y))b_{xy}| \ |v_y| \\
&\leq \varepsilon M \sum_{y\in F} |v_y| \\
&\leq \varepsilon M N_s^{\frac{1}{2}} \ \|v\|.
\end{aligned}
$$

Now $\|av\|^2 = \sum_x |av|_x^2 \leq M^2 N_s^2 \|v\|^2 \varepsilon^2 + \sum_{x\in N_{L^{-1}}} |av|_x^2$, but a being in $C_{2\varepsilon,L}$,

$$\|\chi_F a \chi_{N_{L^{-1}}(F)}\| < \varepsilon$$

hence $\|av\|^2 \leq (M^2 N_s^2 + 1)^{\frac{1}{2}} \|v\|\varepsilon$. □

It remains to prove the lemma.

Proof. Let $b \in C_{\varepsilon,L}$ and $M = \|b\|$. Let $v \in \ell^2(X)$ be a unit vector such that $\|bv\| \leq \|b\| - \frac{\varepsilon}{2M}$ (so that $\|bw\| \geq \|bv\| - \varepsilon$). Denote by μ the probablity measure on X defined by

$$\mu(\{x\}) = |v_x|^2.$$

The MSP implies that there is a subset $\Omega \subset X$ with $\mu(\Omega^c) < \varepsilon$ and Ω is a $\frac{4}{L}$-separated disjoint union

$$\Omega = \coprod_{\frac{4}{L}} \Omega_i$$

of uniformly bounded subsets, i.e. $diam(\Omega_i) < s$ for all i. Denote by $w_i = P_{\Omega_i} v$, and $w = \sum_i w_i$. Then the condition above says that $\|v - w\|^2 < \varepsilon$ and $diam(supp(w_i)) < s$ so if we could approximate $\|b\|$ using one of the w_i's, that would end the proof.

There exists $f_i \in l^\infty(X)_1$ such that

- $Lip(f_i) \leq L$,
- $supp(f_i) \subset N_{L^{-1}}(\Omega_i)$,
- $f_i = 1$ on Ω_i and 0 outside of $N_{L^{-1}}(\Omega_i)$.

Then $f = \sum_i f_i$ and $1 - f$ are also L-lipschitz functions and $fw = w$. But $bw = [b,f]w + fbw$ so

$$
\begin{aligned}
\|bw\| &\leq \varepsilon\|w\| + \|fbfw\| \\
&\leq 2\varepsilon\|w\| + \|\sum_i f_i b f_i w\|
\end{aligned}
$$

In the last line, we used that $\|fbf - \sum f_i b f_i\| \leq \varepsilon$:

Now, the same trick $f_i b = [f_i, b] + bf_i$ entails that

$$\begin{aligned}\|\sum_i f_i b f_i w\|^2 &= \sum_i \|f_i b w\|^2 \\ &\leq \varepsilon \sum_i \|w_i\|^2 + \sum_i \|bw_i\|^2 \\ &\leq \varepsilon\|w\|^2 + \sum_i \|bw_i\|^2\end{aligned}$$

so that

$$(\|bw\| - 3\varepsilon\|w\|)^2 \leq \sum_i \|bw_i\|^2 \leq \sum_i \frac{\|bw_i\|^2}{\|w_i\|^2}\|w_i\|^2 \leq \sup_i(\frac{\|bw_i\|^2}{\|w_i\|^2})\|w\|^2$$

from which follows that

$$\frac{\|bw\|}{\|w\|} \leq \sup_i \frac{\|bw_i\|}{\|w_i\|} + 3\varepsilon.$$

and

$$\sup_i \frac{\|bw_i\|}{\|w_i\|} + 3\varepsilon \geq \frac{\|bv\| - \varepsilon\|v\|}{\|w\|} \geq \|bv\| - \varepsilon \geq \|b\| - 2\varepsilon$$

so that there exists i_0 such that $\frac{\|bw_{i_0}\|}{\|w_{i_0}\|} \geq 6\varepsilon$, and $diam(supp(w_{i_0})) < s$. $\square$

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