C^*-SIMPLICITY, AFTER BREUILLARD, KALANTAR, KENNEDY AND OZAWA

CLÉMENT DELL'AIERA

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These are the notes for the NCG seminar, Fall 2018, of UH Manoa. This semester is devoted to the study of C^*-simplicity for discrete groups, with a focus on the results of Breuillard, Kalantar, Kennedy and Ozawa. These are the references:

- Boundaries of reduced C^*-algebras of discrete groups, Mehrdad Kalantar and Matthew Kennedy [4],
- C^*-simplicity and the unique trace property for discrete groups, Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, Narutaka Ozawa [1],
- An intrinsic characterization of C^*-simplicity, Matthew Kennedy [5].

1. General introduction

Let Γ be a countable discrete group. We will recall two equivalence relations on unitary representations of Γ, which are group homomorphisms

$$\pi : \Gamma \to \mathcal{U}(H_\pi)$$

where $\mathcal{U}(H_\pi)$ stands for the unitary group of a complex Hilbert space H_π. We will refer to such a representation as (π, H_π) or even just π or H_π if no confusion is possible.

Let π and σ be two representations of Γ.

- $\pi \simeq \sigma$ iff there exists a unitary $u : H_\pi \to H_\sigma$ such that
$$u\pi_\gamma u^* = \sigma_\gamma \quad \forall \gamma \in \Gamma.$$
- $\pi \approx \sigma$ iff there exists a sequence of unitaries $u_n : H_\pi \to H_\sigma$ such that
$$\|u_n \pi_\gamma u_n^* - \sigma_\gamma\| \to 0 \quad \forall \gamma \in \Gamma.$$

Fact: It turns out that for a lot of groups (e.g. finite, abelian, compact, simple Lie groups,...), these two notions coincide

$$\pi \approx \sigma \quad \text{iff} \quad \pi \simeq \sigma \quad \text{for } \pi, \sigma \text{ irreducible}.$$

Let $\hat{\Gamma}$ be the collection of all representations of Γ. A very hard problem is to describe

$$\hat{\Gamma} / \approx .$$

It can be done sometimes, e.g. for $\mathbb{Z}$ the irreducible representations are given by the circle, and any representation decomposes more or less uniquely into these.

Let us recall that the (left) regular representation

$$\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$$

is defined by $\lambda_g(\delta_h) = \delta_{gh}$. The reduced C^*-algebra $C_r^*(\Gamma)$ is the closure under the operator norm of the image of the regular representation, i.e.

$$C_r^*(\Gamma) = \overline{span\{\lambda_\gamma\}_{\gamma\in\Gamma}}.$$

A representation π is tempered if it extends to a $*$-representation of $C_r^*(\Gamma)$. This happens iff the linear extension

$$\pi : \mathbb{C}[\Gamma] \to B(H_\pi)$$

satisfies $\|\pi(a)\| \leq \|\lambda(a)\|, \forall a \in \mathbb{C}[\Gamma]$.

Fact: All representations are tempered iff the group is amenable.

Another (very hard) problem is to describe

$$\hat{\Gamma}_r/\approx .$$

Definition 1.1. *Γ is C^*-simple if $C_r^*(\Gamma)$ is simple, i.e. admits no proper two sided closed ideal.*

Theorem 1.2 (Voiculescu). *Γ is C^*-simple iff $\hat{\Gamma}_r/\approx$ is a point.*

Examples of C^*-simple groups:

- Non abelian free groups;
- Torsion free hyperbolic groups;
- $PSL(n, \mathbb{Z})$;
- Thompson's group V.

1.0.1. *Non C^*-simple examples.* Recall that a group Γ is amenable iff the trivial representation

$$1_\Gamma : \Gamma \to \mathcal{U}(\mathbb{C}) = \mathbb{S}^1; \gamma \mapsto id = 1;$$

is tempered. As a consequence, non trivial amenable groups are not C^*-simple as $1 \approx \lambda$ (since $dim(\ell^2\Gamma) \neq 1$ when Γ is not trivial).

More generally if there exists an amenable normal subgroup $K \lhd \Gamma$, then the quasi regular representation

$$\lambda_{\Gamma/K} : \Gamma \to \mathcal{U}(\ell^2(\Gamma/K)); \;\; \lambda_{\Gamma/K}(\gamma)(\delta_{xK}) = \delta_{\gamma xK};$$

is tempered, hence if K is not trivial, Γ is not C^*-simple. In particular any semi-direct product $K \rtimes H$ with K amenable and non trivial is not C^*-simple.

Amenability being stable by extensions and increasing unions, any group has a largest normal amenable subgroup $R \lhd \Gamma$ called the amenable radical. The previous discussion shows that if Γ is C^*-simple, then $R = \{e\}$. The converse does not hold and was completely answered by Kennedy et al.

1.0.2. *How to prove C^*-simplicity?* À la Powers [7].

Definition 1.3. *A group Γ is a Powers group if for every finite subset $F \subset \Gamma$ there exists a partition*

$$\Gamma = C \coprod D$$

and a finite number of elements $\gamma_1, ..., \gamma_n \in \Gamma$ with

- *$\gamma C \cap C = \emptyset$ for every $\gamma in F$:*
- *$\gamma_i D \cap \gamma_j D = \emptyset$ for every $i \neq j$.*

Examples:

- The free group on two generators $\mathbb{F}_2$ (Powers [7]);
- Many other examples using"North-South" type dynamics (De la Harpe, Bridson, Osin).

Let us write a few words about the technique Powers used. For $\mathbb{F}_2 = \langle a, b\rangle$, let

$$\tau : C_r^*(\mathbb{F}_2) \to \mathbb{C}; a \mapsto \langle \delta_e, a\delta_e \rangle$$

be the canonical tracial state.

Theorem 1.4 (Powers [7])**.** *For every $a \in C_r^*(\Gamma)$,*

$$\tau(x) = \lim \frac{1}{mn} \sum_{i=1,n \ j=1,m} \lambda_a^i \lambda_b^j x \lambda_b^{-j} \lambda_a^{-i}$$

Corollary 1.5. *$\mathbb{F}_2$ is C^*-simple.*

Proof. Let $J \lhd C_r^*(\mathbb{F}_2)$ be an ideal. For $x \in C_r^*(\mathbb{F}_2)$ let

$$x_{mn} = \sum_{i=1,nj=1,m} \lambda_a^i \lambda_b^j x \lambda_b^{-j} \lambda_a^{-i}.$$

If $x \in J$ then $(x^*x)_{mn} \in J$ so $\tau(x^*x)1_{C_r^*(\mathbb{F}_2)} \in \overline{J}^{\| \ \|}$. If J is not trivial, it contains a non zero element x, which forces $1_{C_r^*(\mathbb{F}_2)} \in J$ as $\tau(x^*x) > 0$. This ensures that $J = C_r^*(\mathbb{F}_2)$ and we are done. □

Corollary 1.6. *$C_r^*(\mathbb{F}_2)$ has a unique tracial state.*

Proof. Let τ' be a tracial state on $C_r^*(\mathbb{F}_2)$. Then for $x \in C_r^*(\mathbb{F}_2)$,

$$\tau'(x) = \tau'(x_{mn}) \to \tau'(\tau(x)1) = \tau(x)\tau'(1) = \tau(x).$$

□

2. Definitions

We only consider discrete countable groups, usually denoted by Γ.

Definition 2.1. *A group is said to be C^*-simple if its reduced C^*-algebra is simple, i.e. has no proper closed two sided ideals.*

A motivation for the interest toward such a notion can be the following result of Murray and Von Neumman: the Von Neumman algebra $L(\Gamma)$ is simple (no proper weakly closed two sided ideals) iff it is a factor iff Γ is ICC (infinite conjugacy classes, i.e. all non trivial conjugacy classes are infinite). Another one is that simplicity is one out of the 5 criteria (unital simple separable UCT with finite nuclear dimension) needed in Elliott's classification programm (see for instance [3] for a good introduction).

Recall that, given two unitary representations of Γ, we say that π is weakly contained in σ and write

$$\pi < \sigma$$

if every positive type function associated to π can be approximated uniformly on compact sets by finite sums of such things associated to σ. In other words, if for every $\xi \in H_\pi$, every $F \subseteq \Gamma$ finite and every $\varepsilon > 0$, there exists $\eta_1, \eta_2, ..., \eta_k$ such that

$$|\langle \pi(s)\xi, \xi\rangle - \sum_i \langle \sigma(s)\eta_i, \eta_i\rangle| < \varepsilon \quad \forall s \in F.$$

Remark: one can restricts to convex combinations of normalized positive type functions.

If $\pi < \sigma$, then $id_{\mathbb{C}[\Gamma]}$ extends to a surjective $*$-homomorphism

$$C^*_\sigma(\Gamma) \to C^*_\pi(\Gamma).$$

Indeed, it suffices to show that for every $a \in \mathbb{C}[\Gamma]$,

$$\|\pi(a)\| \leq \|\sigma(a)\|.$$

As $\|\pi(a)\|^2 = \|\pi(a^*a)\|$, we can suppose a positive. Then

$$\begin{aligned} \langle \pi(s)\xi, \xi\rangle &\leq \sum_i t_i \langle \sigma(s)\eta_i, \eta_i\rangle + \varepsilon \\ &\leq \|\sigma(a)\| + \varepsilon \end{aligned}$$

hence $\|\pi(a)\| \leq \|\sigma(a)\| + \varepsilon$, and let just ε go to zero.

Definition 2.2. *A group Γ is C^*-simple if its reduced C^*-algebra is simple (i.e. has no proper closed two sided ideal).*

Theorem 2.3. *If Γ has a non trivial amenable normal subgroup, then it is not C^*-simple.*

Proof. *Let N be a normal amenable subgroup of Γ. Let (F_k) be a sequence of Følner sets for N, and*

$$\xi_k = \frac{1}{|F_k|^{\frac{1}{2}}} \chi_{F_k} \in \ell^2(\Gamma)$$

Then

$$\|s \cdot \xi_k - \xi_k\|_2^2 = 2\left(1 - \Re\langle \lambda_\Gamma(s)\xi_k, \xi_k\rangle\right) = 2\left(1 - \frac{|F_k \Delta s F_k|}{|F_k|}\right).$$

The term $\frac{|F_k \Delta s F_k|}{|F_k|}$ is 0 if $s \notin N$, and goes to 1 as n goes to infinity if $s \in N$, hence

$$\langle \lambda_\Gamma(s)\xi_k, \xi_k\rangle \to \langle \lambda_{\Gamma/N}(s)\delta_{eN}, \delta_{eN}\rangle,$$

which shows that $\lambda_{\Gamma/N} < \lambda_\Gamma$. This gives us a surjective $*$-morphism

$$\phi : C^*_r(\Gamma) \to C^*_{\Gamma/N}(\Gamma).$$

A faster but more involved argument, which still works out when the ambient group is only locally compact, is the following. As N is amenable,

$$1_N < \lambda_N,$$

ensures by induction

$$Ind_N^\Gamma 1_N = \lambda_{\Gamma/N} < Ind_N^\Gamma \lambda_N = \lambda_\Gamma.$$

But if $n \in N$ is non trivial, $\lambda_\Gamma(n)$ is non trivial and sent to $\lambda_{\Gamma/N}(n) = 1$ via ϕ, so that $Ker\ \phi$ is a proper ideal in $C^*_r(\Gamma)$.

□

After the talk, Erik Guentner suggested the following proof. It is even shorter and doesn't assume any knowledge about weak containment or induction of representations. It is a weakening of the following fact: when Γ is amenable, the trivial representation $1_\Gamma : C^*_{max}(\Gamma) \to \mathbb{C}$ extends to the reduced C^*-algebra.

Indeed let $a \in \mathbb{C}[\Gamma]$ and (F_n) be a sequence of Følner sets for Γ. Define $\xi_n = \frac{1}{|F_n|^{\frac{1}{2}}} \chi_{F_n} \in \ell^2(\Gamma)$. Then, suppose a is positive, and compute

$$\begin{aligned} \langle a\xi_n, \xi_n \rangle &= \sum_{s \in \text{ supp } a} a_s \frac{|F_n \cap sF_n|}{|F_n|} \\ &\to \|a\|_{1_\Gamma} = \sum_s a_s \end{aligned}$$

so that $\|a\|_{1_\Gamma} \leq \|a\|_r$.

Now if N is a normal amenable subgroup of Γ, do the same with Følner sets for N, and the coefficients of the induced representation $\lambda_{\Gamma/N}$.

Remark that both conditions are necessary. Indeed, we saw that $\mathbb{F}_2$ is C^*-simple, yet it has a copy of $\mathbb{Z}$ as an amenable subgroup (non normal), and a normal (non amenable) subgroup: the commutator subgroup, which is an infinite rank free group, $\langle [x, y] : x, y \in \mathbb{F}_2 \rangle = \mathbb{F}([a^n, b^m]; n, m)$.

This result led to following (false) conjecture: a group is C^*-simple iff it has no non trivial amenable normal subgroups.

3. Injective C^*-algebras

Recall that an abelian group M is injective if, given any injective homomorphism of abelian group $A \hookrightarrow B$, any homomorphism $A \to M$ extends to a homomorphism $B \to M$. In words: any homomorphism into M extends to super-objects. We will often use the following commutative diagram

$$\begin{array}{ccc} B & & \\ \uparrow & \overset{\exists}{\searrow} & \\ A & \longrightarrow & M \end{array}$$

to represent this situation. We will now turn to an analog notion in the C^*-algebraic setting.

Definition 3.1. *A C^*-algebra M is injective if, given an inclusion of C^*-algebras $A \subset B$, any injective $*$-homomorphism $A \to M$ extends to B by a contractive completely positive (CCP) map.*

$$\begin{array}{ccc} B & & \\ \uparrow & \overset{\exists\ ccp}{\searrow} & \\ A & \longrightarrow & M \end{array}$$

Even if the straight arrow are here supposed to be $*$-homomorphism, Stinespring's dilation theorem ensures that we can suppose all the arrows to be only CCP maps.

We will say that M is Γ-injective if Γ acts by automorphisms on all the C^*-algebras in the diagram, and all the arrows are Γ-equivariant.

We will define a particular class of compact spaces acted upon by Γ, called Γ-boundaries, and show that there exists a maximal Γ-boundary $\partial_F\Gamma$, called the *Furstenberg boundary.*

The first **major goal** of this presentation is to show that $C(\partial_F\Gamma)$ is Γ-injective.

3.0.1. *Description of commutative injective algebras.*

Lemma 3.2. *If M is injective and $S \subset M$, define*

$$Ann_M(S) = \{m \in M \mid \forall s \in S,\ sm = 0\}.$$

Then there exists a projection $p \in M$ satisfying $Ann_M(S) = pM$.

Proof. This is true if $M = \mathcal{B}(H)$ for some Hilbert space. In the general case, embed M unitally in some $\mathcal{B}(H)$. By injectivity of M, there exists a CCP map $E : \mathcal{B}(H) \to M$ such that $E(m) = m, \forall m \in M$ so $M \subset dom(E)$ (multiplicative domain). There exists a projection $p \in \mathcal{B}(H)$ with $Ann_{\mathcal{B}(H)}(S) = p\mathcal{B}(H)$ (take the projection on $\cap_{s\in S} Ker(s)$). If $s \in S$,

$$sE(p) = E(sp) = 0 \text{ hence } E(p) \in Ann_M(S).$$

Moreover if $m \in Ann_M(S) \subset p\mathcal{B}(H)$, $pm = m$ and

$$E(p)m = E(pm) = E(m) = m$$

so that for $m = E(p)$, we get $E(p)$ is a projection. This also proves that

$$E(p)Ann_M(S) = Ann_M(S).$$

A slight fiddling ensures then that $Ann_M(S) = E(p)M$. □

Corollary 3.3. *Let X be a compact Hausdorff space. If $C(X)$ is injective then X is Stonean, i.e. $\overline{U}$ is open for every open subset $U \subset X$.*

Proof. Let $U \subset X$ be open, and $S = C_0(U)$. By the previous lemma, there exists a projection $p \in C(X)$ such that $Ann_{C(X)}(S) = pC(X)$. But p cannot be anyone else than the characteristic function of $\overline{U}^c$ so that $1 - p = \chi_{\overline{U}}$ is continuous and $\overline{U}$ is open. □

Note: Infinite compact Stonean spaces are not metrizable (not even second countable). Suppose the contrary and get a sequence $x_i \to x$ in X and open sets $U_n = B(x_n, \varepsilon_n)$, with ε_n such that $\overline{U}_n \cap \overline{U}_m = \emptyset$ for every $n \neq m$. Set $U = \cup_n U_{2n}$, then $x \in \overline{U}$ ($\overline{U}$ is open) so $x_n \in \overline{U}$ for large n but $x_n \notin \overline{U}$ for n odd.

4. Furstenberg boundary

If Γ is a discrete group acting on a compact Hausdorff space X (we will just say that X is a Γ-space), the space of probability measures $Prob(X)$ endowed with the weak-$*$ topology is homeomorphic to the state space $S(C(X))$ with the topology of simple convergence. We identify X with a closed subspace of $Prob(X)$ with the help of the Dirac masses ($x \mapsto \delta_x$ is an embedding $X \hookrightarrow Prob(X)$). Recall that the action can be extended to $Prob(X)$, which is then a Γ-space by Banach-Alaoglu's theorem.

Definition 4.1. *A Γ-space X is:*

- *minimal if the only Γ-invariant closed subset of X are itself and $\emptyset$;*
- *strongly proximal if $\overline{\Gamma.\mu}^{weak-*}$ contains δ_x for some $x \in X$;*

- *a* Γ*-boundary if it is minimal and strongly proximal*

$$X \subset \overline{\Gamma.\mu}^{weak-*} \quad \forall \mu \in Prob(X).$$

Example: Let $SL(2,\mathbb{Z})$ act on the projective line $\mathbb{R}P^1$ (the quotient of $\mathbb{R}^2\backslash\{0\}$ by the group of dilations) given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

Most $g \in Sl(2,\mathbb{Z})$ are acting hyperbolically (two distinct eigenspaces, one expansive one contractive). Take $\mu \in Prob(\mathbb{R}P^1)$, and a generic element $g \in SL(2,\mathbb{Z})$. As n goes to ∞,

$$g^n.\mu \to_{weak-*} \delta_{\text{Expanding eigenspace}}$$

unless $\mu(\{\text{contractive eigenspace}\}) > 0$, hence

$$\{\delta_{\text{Expanding eigenspace}}\}_{g\in SL(2,\mathbb{Z})} \subset \overline{\Gamma.\mu}^{wk-*}.$$

Exercise: the set of these is dense in $\mathbb{R}P^1 \subset Prob(\mathbb{R}P^1)$.

Theorem 4.2 (Furstenberg [2])**.** *There exists a* Γ*-boundary* $\partial_F\Gamma$ *(now called the Furstenberg boundary) such that for any* Γ*-boundary* X *there exists a continuous* Γ*-equivariant surjection* $\partial_F\Gamma \twoheadrightarrow X$*.*

Proof. Let $\mathcal{B}$ be the class of all Γ-boundaries. It is non empty as it contains the point space. Take

$$Z = \prod_{Y\in\mathcal{B}} Y$$

which is compact by Tychonoff's theorem. Equip Z by the diagonal Γ-action.

- It is strongly proximal: for any $\mu \in Prob(Z)$, a diagonal argument gives a weak-$*$ convergent net $g_i.\mu \to \delta_z$ for some $z \in Z$.
- It is not minimal, but Zorn's lemma ensures the existence of a minimal closed Γ-invariant subset $\partial_F\Gamma$ of Z.

We obtain the desired map as the composition of the inclusion $\partial_F\Gamma \hookrightarrow Z$ with the projection on the X-factor $Z \twoheadrightarrow X$. ☐

Theorem 4.3 (Kalantar-Kennedy [4])**.** $C(\partial_F\Gamma)$ *is* Γ*-injective.*

Lemma 4.4. *There exists a bijective correspondence between the completely positive maps from* $C(X)$ *to* $C(Y)$ *and the continous maps from* Y *to* $Prob(X)$*. The statement remains true if one asks for equivariance.* ***send to a previous section on CP maps***

Lemma 4.5 (Furstenberg)**.** *Let* X *and* Y *be two* Γ*-boundaries. Then any* Γ*-equivariant map* $X \to Prob(Y)$ *has image in* Y*, i.e. any UCP map* $C(X) \to C(Y)$ *is a* $*$*-homomorphism! Moreover there is at most one such map.*

Proof. Take $\mu : X \to Prob(Y)$. The image $\mu(X) \subset Prob(Y)$ is a closed Γ-invariant subspace: by strong proximality of Y, there exists $y \in Y$ such that

$$\delta_y \in \overline{\Gamma.\mu_x}^{wk-*} \subset \mu(X).$$

By minimality of Y, $\overline{\Gamma.\mu_x}^{wk-*} \cap Y = Y$, By minimality of X, $\mu^{-1}(Y) = X$ i.e. $\mu(X) \subset Y$.

Let $\mu, \eta : X \to Prob(Y)$ be two such maps. Then $\frac{1}{2}\mu + \frac{1}{2}\eta$, μ and η all take values in Y so that they are all equal. ☐

Corollary 4.6. *Any equivariant UCP map* $C(\partial_F\Gamma) \to C(\partial_F\Gamma)$ *is the identity.*

Recall that if A is a unital Γ-algebra, its state space $S(A)$ is convex compact Γ-space.

Proposition 4.7 (Gleason)**.** *Let $Z \subset S(A)$ be a Γ-invariant closed convex subspace, which is minimal w.r.t. these properties. (Such a thing exists by Zorn's lemma.) Then*

$$\partial_{ex} Z = \{\phi \in Z \mid \phi \textit{ is not a non trivial convex combination of anything in } Z\}$$

is a Γ-boundary.

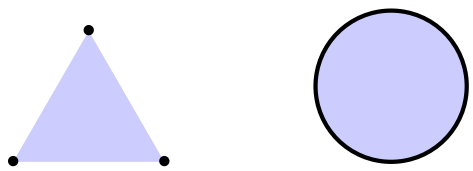


FIGURE 1. Two examples with Z in blue and $\partial_{ex} Z$ in black.

Proof. There is a barycenter map $\beta : Prob(Z) \to Z$ such that

$$\int_Z f d\mu = f(\beta(\mu)) \quad \forall f \in C(Z) \text{ affine.}$$

Indeed, if $\mu = \delta_z$, $\beta(\mu) = z$ and if $\mu = \sum \alpha_i \delta_{z_i}$ with $0 \leq \alpha_i \leq 1$ and $\sum \alpha_i = 1$, then $\beta(\mu) = \sum \alpha_i z_i$. Finite convex combinations are weak-$*$ dense in $Prob(Z)$ by the Hahn-Banach separation theorem. As β is weak-$*$ continuous, and affine so uniformly weak-$*$ continuous, it extends to the whole space $Prob(Z)$.

Note: β is Γ-equivariant continuous and satisfies $\beta(\mu) = z \in \partial_{ex} Z$ iff $\mu = \delta_z$.

Then, for any $\mu \in Prob(Z)$,

$$\beta(\overline{conv(\Gamma\mu)}) = \overline{conv(\Gamma\beta(\mu))} = Z,$$

the first equality coming from continuity, Γ-equivariance and affinity. Now, $\partial_{ex} Z$ is minimal, and if $\mu \in \partial_{ex} Z$, then

$$AFINIR$$

□

We are now ready for the main result of this section.

Theorem 4.8 (Kalantar-Kennedy)**.** *$C(\partial_F \Gamma)$ is Γ-injective.*

Proof. First, observe that $\ell^\infty(\Gamma)$ is Γ-injective. Let indeed $A \subset B$ be an inclusion of C^*-algebras and $\phi : A \to \ell^\infty(\Gamma)$ a $*$-homomorphism. Then $ev_e \circ \phi$ is a state on A, so it extends to a state Ψ on B. Define $\tilde{\phi} : B \to \ell^\infty(\Gamma)$ by

$$\tilde{\phi}(b)(\gamma) = \Psi(\gamma^{-1}.b).$$

Then Ψ is a UCP Γ-equivariant map that extends ϕ.

Now, producing ucp equivariant maps

$$C(\partial_F \Gamma) \xrightarrow{\alpha} \ell^\infty(\Gamma) \xrightarrow{\beta} C(\partial_F \Gamma)$$

is sufficient to conclude, as their composition must be the identity by corollary 4.6.

Define $\alpha : C(\partial_F\Gamma) \to \ell^\infty(\Gamma)$ by fixing $\mu \in Prob(\partial_F\Gamma)$ and set

$$\alpha(f)(\gamma) = \mu(\gamma^{-1}.f).$$

By Gleason's theorem 4.7, there is a Γ-boundary $X \subset S(\ell^\infty(\Gamma))$. By universal property of $\partial_F\Gamma$, we have an equivariant surjection $\partial_F\Gamma \twoheadrightarrow X \subset S(\ell^\infty(\Gamma))$. By duality, we get a Γ-equivariant ucp map

$$\Psi : \ell^\infty(\Gamma) \to C(\partial_F\Gamma)$$

and we are done. □

As a final remark, one can point out that this last proof used the following useful fact: if B is injective and $\phi : A \to B$ is a split injective Γ-ucp map, then A is injective. We use this with $A = C(\partial_F\Gamma)$ and $B = \ell^\infty(\Gamma)$.

5. Dynamical characterization of C^*-simplicity

We will be using the following facts:

- $C(\partial_F\Gamma)$ is Γ-injective, in particular any Γ-equivariant u.c.p. $C(\partial_f\Gamma) \to A$ is split, so is an isometric embedding,
- $\partial_F\Gamma$ is totally disconnected.

The goal of this section is to prove the following theorem.

Theorem 5.1. *Γ is C^*-simple iff the action of Γ on $\partial_F\Gamma$ is free.*

Let's do first the forward direction.

Suppose the action is free. First, to show $C^*_r(\Gamma)$ is simple, it is enough to show that any representation

$$\pi : C^*_r(\Gamma) \to \mathcal{B}(H)$$

is injective.

By Arveson's extension theorem, π extends to a u.c.p. map

$$\phi : C(\partial_F\Gamma) \rtimes_r \Gamma \to \mathcal{B}(H).$$

Its restriction ϕ_0 to $C(\partial_F\Gamma)$ is Γ-equivariant, because $C(\partial_F\Gamma)$ is in the multiplicative domain of ϕ_0, and thus must be an isometric embedding, by Γ-injectivity of $C(\partial_F\Gamma)$ (it is split because $\mathbb{C} \subseteq \mathcal{B}(H)$). The equivariant u.c.p. map ϕ_0 is an isomorphism onto its image: extend its inverse form $im\ \phi_0$ to $im\ \phi$ and denote the resulting u.c.p map by τ.

Claim: $\Psi = \tau \circ \phi$ is the canonical expectation $E : C(\partial_F\Gamma) \rtimes_r \Gamma \to C(\partial_F\Gamma)$ which is faithful. This implies π is injective.

Let's end up with the claim.

- $\Psi_{|C(\partial_F\Gamma)} = id_{C(\partial_F\Gamma)}$. Indeed, τ is the inverse of $\phi_0 = \phi_{C(\partial_F\Gamma)}$.
- If $\gamma \neq e_\Gamma$, the action being free, for every x there exists a function $f \in C(\partial_F\Gamma)$ such that

  $$f(x) \neq 0 \quad \text{and} \quad f(s^{-1}x) = 0.$$

  Now $C(\partial_F\Gamma)$ is in the multiplicative domain of Ψ, so

  $$\Psi(\lambda_s)f = \Psi(\lambda_s f) = \Psi((sf)\lambda_s) = (sf)\Psi(\lambda_s)$$

  which evaluated at x gives $\Psi(\lambda_s)(x) = 0$, for all x, so $\Psi(\lambda_s) = 0$.

The other direction is more intricated. It consists in two steps:

(1) if $x \in \partial_F \Gamma$, then the stabilizer Γ_x is amenable, which implies that $\lambda_{\Gamma/\Gamma_x} < \lambda_\Gamma$,

(2) if X is a X is a Γ-boundary, and $\gamma \neq 0$ such that $int(X_s) \neq \emptyset$, then $\lambda_\Gamma \not< \lambda_{\Gamma/\Gamma_x}$, so that the kernel of $C^*_r(\Gamma) \to C^*_{\lambda_{\Gamma/\Gamma_x}}(\Gamma)$ is a non trivial two sided closed ideal.

This, together with the fact that $\partial_F \Gamma$ is topologically free iff it is free, concludes the proof.

First bullet:

- there exists a Γ_x-equivariant injective $*$-homomorphism
$$\rho : \ell^\infty(\Gamma_x) \to \ell^\infty(\Gamma)$$
defined by $\rho(f)(ts_i) = f(t)$ for every $t \in \Gamma_x$, $\{s_i\}_i$ being a system of representatives of the right cosets $\Gamma_x \backslash \Gamma$.

- there exists a Γ_x-equivariant u.c.p. map
$$\psi : \ell^\infty \to C(\partial_F \Gamma),$$
by universal property of $\partial_F \Gamma$, and the fact that the spectrum of $\ell^\infty(\Gamma)$ is $\beta\Gamma$. (for any compact Γ-space, there exists a Γ-map $\partial_F \Gamma \to P(X)$. take the dual of this map for $X = \beta\Gamma$).

- The composition $\phi = ev_x \circ \psi \circ \rho$ defines a Γ_x-invariant state on $\ell^\infty(\Gamma_x)$, which concludes the proof.

Second bullet:

This needs a lemma:

Lemma 5.2. *Let X be a Γ-boundary. For every non empty subset of X, every $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma \backslash \{e_\Gamma\}$ such that*
$$\min_{t \in F} \mu(tU^c) < \varepsilon \quad \forall \mu \in P(X).$$

Proof. Let $x \in U$. By strong proximality, there exists $t_\mu \neq e_\Gamma$ such that
$$\delta_x(U) - \mu(t_\mu U) = \mu(t_\mu U^c) < \varepsilon,$$
and by continuity of the action
$$V_\mu = \{\nu \in P(X) \mid \nu(t_\mu U^c) < \varepsilon\}$$
is a neighboorhood of μ. By compactness of $P(X)$ in the weak-$*$ topology, we can extract a finite cover such that
$$P(X) = \cup_{i=1,m} V_{\mu_i}.$$
Then $F = \{t_{\mu_1}, ..., t_{\mu_m}\}$ fills the requirements of the lemma.

□

Suppose the action is not topologically free and let $s \neq e_\Gamma$ such that the interior U of X_s is not empty. Let F the finite subset given by the lemma for U and $\varepsilon = \frac{1}{3}$. Suppose

$$\lambda_\Gamma < \lambda_{\Gamma/\Gamma_x}.$$

We will show this is absurd by looking at the coefficient $c_\gamma = \langle \lambda_\Gamma(\gamma)\delta_e, \delta_e \rangle$, which is 0 unless $\gamma = e_\Gamma$.

On the finite subset $K = \{tst^{-1}\}_{t \in F}$, approximate c_γ up to ε by a convex combination

$$\sum_{j=1,n} \alpha_j \langle \lambda_{\Gamma/\Gamma_x}(\gamma)\xi_j, \xi_j \rangle$$

of coefficients of the quasi regular representation. Set

$$\mu_j = \sum_{y \in \Gamma.x} |\xi_j(y)|^2 \delta_y \in P(X) \quad \text{and} \quad \mu = \sum \alpha_j \mu_j,$$

where we identify $\Gamma.x$ with Γ/Γ_x. **A FINIR**

Questions:

- Can we get a more direct proof for the last implication? (without representation theory)
- It is not known in general wether the action of Γ on $\partial_F \Gamma$ is amenable. If X is a Γ-space such that one of the stabilizer is not amenble, the action cannot be amenable. Is it true that, if Γ is exact, this is the only obstruction for the amenability of the action?

6. Another proof

The last subsection uses representation theory (induction) which makes one wonder if this could be avoided. While the implication

$$\partial_F \Gamma \text{ is free } \Rightarrow \Gamma \text{ is } C^*\text{-simple}$$

is still good enough if one wants to stay clear of representation theoretic lingo, the other direction can be proven in another way.

This proof is taken from a set of notes that Ozawa wrote after giving lectures for the "Annual Meeting of Operator Theory and Operator Algebras" at Tokyo university, 24–26 December 2014.

For X a compact Γ-space and H a subgroup of Γ, we denote by:

- $E_x : C(X) \rtimes_r \Gamma \to C^*_r(\Gamma)$ the canonical conditional expectation onto $C^*_r(\Gamma)$ given by extending the evaluation at x,
- $E_H : C^*_r(\Gamma) \to C^*_r(H)$ the canonical conditional expectation given by $E(\lambda_s) = \delta_{s \in H}$,
- τ_H the canoncical trace $C^*_r(H) \to \mathbb{C}$.

The first thing one can show is the following.

Proposition 6.1. *Let X be a Γ-boundary, then*

$$C(X) \rtimes_r \Gamma$$

is simple.

Proof. It is enough to show that any quotient map

$$\pi : C(X) \rtimes_r \Gamma \to B$$

is injective. By C^*-simplicity, π restricts to an isomorphism on $C_r^*(\Gamma)$ so that the canoncial trace τ is well defined on $\pi(C_r^*(\Gamma))$. Seeing $\mathbb{C}$ as the sub-C^*-algebra of constant functions in $C(\partial_F \Gamma)$, we can extend τ to B.

$$\begin{array}{ccccc} C(X) \rtimes_r \Gamma & \xrightarrow{\pi} & B & \overset{\phi}{\dashrightarrow} & \\ \uparrow & & \uparrow & & \\ C_r^*(\Gamma) & \xrightarrow{\cong} & \pi(C_r^*(\Gamma)) & \xrightarrow{\tau} & \mathbb{C} \subseteq C(\partial_F \Gamma) \end{array}$$

Now $\phi \circ \pi$ restricts to a Γ-u.c.p. map $C(X) \to C(\partial_F \Gamma)$ which can only be the inclusion. This ensures that

$$C(X) \subseteq Dom(\phi \circ \pi).$$

As ϕ extends τ, $\phi \circ \pi$ is the canonical conditional expectation $C(X) \rtimes_r \Gamma \to C(X)$ which is faithful. In particular, π is faithful, and is injective. $\square$

Applying this to $X = \partial_F \Gamma$, we get that $C(\partial_F \Gamma) \rtimes_r \Gamma$ is simple. In that case, every stabilizer

$$\Gamma_x = \{s \in \Gamma \mid sx = x\} \quad \forall x \in \partial_F \Gamma$$

is amenable. Moreover, the strong stabilizer

$$\Gamma_x^0 = \{s \in \Gamma \mid \exists U \text{ neighborhood of } x \text{ s.t. } s_U = id_U\}$$

is a normal subgroup of Γ_x.(In particular, is is amenable.) In that case, we will apply the following proposition.

Proposition 6.2. *Let X be a minimal compact Γ-space. If*

$$C(X) \rtimes_r \Gamma$$

is simple and there exists $x \in X$ such that Γ_x^0 is amenable, then X is topologically free.

Proof. By minimality, topological freeness is equivalent to $\Gamma_x^0 = 1$ for some x.

Indeed, if $\Gamma_x^0 = 1$ for some x, every non trivial group element cannot fix any neighborhood of x hence for every $s \neq e_\Gamma$, we get a sequence of points that converge to x which are not fixed by s. By minimality,

$$X_s = \{y \in X \mid sy \neq y\}$$

is a non empty dense open set of X for every $s \neq e_\Gamma$. By Baire category's theorem,

$$\cap_{s \in \Gamma \setminus \{e\}} X_s$$

is dense in X so that X is topologically free.

Let us show that $\Gamma_x^0 = 1$. Define a representation

$$\rho : C(X) \rtimes_r \Gamma \to B(\ell^2(\Gamma/\Gamma_x^0))$$

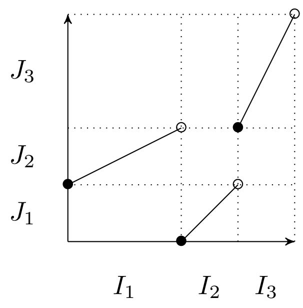


FIGURE 2. The graph of $\begin{pmatrix} I_1 & I_2 & I_3 \\ J_2 & J_1 & J_3 \end{pmatrix}$

by $\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = f(s\gamma.x)\delta_{s\gamma\Gamma_x^0}$. It is clearly covariant on the algebraic crossed-product.

To prove ρ extends to the whole crossed-product, i.e. $\|\rho(a)\| \leq \|a\|_{C(X)\rtimes_r\Gamma}$, it is enough to show that

$$\langle\ \rho(a)\delta_{\Gamma_x^0}, \delta_{\Gamma_x^0}\ \rangle \leq \|a\|_{C(X)\rtimes_r\Gamma}$$

because $\delta_{\Gamma_x^0}$ is cyclic. This follows from the fact that the latter is the composition $\tau \circ E_{\Gamma_x^0} \circ E_x$ of 3 u.c.p maps (so contractive).

Pick up x such that Γ_x^0 is amenable and $s \in \Gamma$ arbitrary that fixes some neighborhood of x: there exists a neighborhood U of x such that $s_{|U} = id_U$. Let $f \in C(X)$ be nonzero and supported in U. Let us compute

$$\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0}.$$

- If $\gamma.x \in U$, then $s\gamma.x = \gamma.x$ and
$$\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = f(\gamma.x)\delta_{\gamma\Gamma_x^0} = \rho(f)\delta_{\gamma\Gamma_x^0}.$$
- If $\gamma.x \notin U$, $f(\gamma.x) = 0 = f(s\gamma.x)$, so that $\rho(f\lambda_s)\delta_{\gamma\Gamma_x^0} = \rho(f)\delta_{\gamma\Gamma_x^0}$.

This shows that $\rho(f(\lambda_s - 1)) = 0$. By injectivity, $\lambda_s = 1$ and $s = e_\Gamma$ hence $\Gamma_x^0 = 1$ and we are done. $\square$

7. THOMPSON'S GROUP V IS C^*-SIMPLE

In this section, we prove that Thompson's group V is C^*-simple. Recall that V is defined as the group of piecewise linear bijections of $[0, 1)$ with finitely many points of non differentiability, all of which are dyadic rational numbers. Such a function f is entirely determined by two partitions

$$[0,1) = \coprod_{i=1}^{n} I_i = \coprod_{i=1}^{n} J_i$$

and a bijection $\begin{pmatrix} I_1 & \dots & I_n \\ J_{\sigma(1)} & \dots & J_{\sigma(n)} \end{pmatrix}$. The intervals I_i and J_i are of the type $[a, a + 2^{-n})$, with a dyadic rational in $[0, 1)$. Then f is defined on I_i as the only linear increasing function applying I_i to $J_{\sigma(i)}$.

In order to prove that V is C^*-simple, we will:

- realize V as a countable group of homeomorphisms of the Cantor set;

- use the following result of Le Boudec and Matte-Bon ([6], thm 3.7):

Theorem 7.1. *Let X be a Hausdorff locally compact space and Γ be a countable subgroup of $Homeo(X)$. Suppose that for every non empty open subset $U \subset X$, the rigid stabilizer*
$$\Gamma_U = \{\gamma \in \Gamma \mid \gamma x = x \ \forall x \notin U\}$$
is non amenable. Then Γ is C^*-simple.

Let G be an ample groupoid with compact base space. We also always suppsose that groupoids are second countable, Hausdorff and locally compact. Recall that a bisection $U \subset G$ is a set such that s and r are homoeomorphisms when restricted to U. In particular, any open bisection U induces a partial homeomorphism
$$\alpha_U \left\{ \begin{array}{rcl} s(U) & \to & r(U) \\ x & \mapsto & r \circ s_{|U}^{-1}(x) \end{array} \right.$$
The topological full group $[\![G]\!]$ is defined as the set of bisections U of G such that $s(U) = r(U) = G^0$. The operations are defined by
$$e = G^0, \quad UV = \{gg' \mid g \in U, g' \in V \text{ s.t. } s(g) = r(g')\}, \quad U^{-1} = \{g^{-1} \mid g \in U\}.$$
Recall that a Cantor space is any compact metrizable totally disconnected space without any isolated points. It is a standard fact that they are all homeomorphic. A model for Ω is the countable product A^X, where

- A is a finite set, often reffered to as the *alphabet*;
- X is a countable set.

Then elements of Ω are infinite words indexed by X. Denote by Ω_f the set of finite words
$$\Omega_f = \coprod_{\text{finite } F \subset X} A^F,$$
then the topology on Ω is the one generated by the *cylinders*
$$C_a = \{w \in \Omega \mid w(x) = a(x) \ \forall x \in F = supp(a)\}.$$
For finite words $a \in \Omega_f$, $l(a)$ denotes their length, and if $F = \mathbb{N}$, $x \in \Omega$, ax denotes the concatenation of a and x, i.e. the word obtained by first saying a and then x.

Examples:

(1) Let Γ a countable discrete group acting on a Hausdorff compact space X by homeomorphisms. Then $[\![X \rtimes \Gamma]\!]$ consists of the bisections of the type
$$S = \coprod U_i \times \{\gamma_i\}$$
where $X = \coprod_{i=1}^n U_i = \coprod_{i=1}^n \gamma_i U_i$.

(2) Let $\mathbb{Z}$ act on the Cantor space $\Omega = \{0,1\}^{\mathbb{Z}}$ by Bernoulli shift
$$n(a_i)_i = (a_{i+n})_i \quad \forall n \in \mathbb{Z}, a \in \Omega.$$
Then $[\![\Omega \rtimes \mathbb{Z}]\!]$ consists of homeomorphisms $\phi : \Omega \to \Omega$ such that there exists a continuous function $n : \Omega \to \mathbb{Z}$ such that
$$\phi(x) = n(x).x \quad \forall x \in \Omega.$$

(3) Let $\Omega = \{0,1\}^{\mathbb{N}}$ be another model for the Cantor space. Define $T : \Omega \to \Omega$ continuous to be the shift
$$T(a_0, a_1, ...) = (a_1, a_2, ...).$$
Let G_2 be the so-called *Cuntz* or *Renault-Deaconu* groupoid defined by
$$\{(x, m-n, y) \mid x, y \in \Omega, m, n \in \mathbb{N} \text{ s.t. } T^m x = T^n y\}.$$

Exercise: The reduced C^*-algebra of G_2 is isomorphic to the Cuntz algebra

$$O_2 = C^*\langle s_1, s_2 \mid s_1 s_1^* + s_2 s_2^* = 1, s_1^* s_1 = s_2^* s_2 = 1\rangle.$$

The open sets

$$U_{a,b} = \{(ax, l(a) - l(b), bx) \mid x \in \Omega\}$$

define compact open bisections which cover G_2 when a, b run across Ω_f.

Then $[\![G_2]\!]$ consists of the bisections of the type

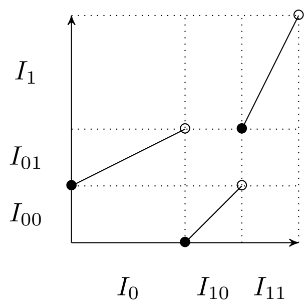
$$S = \coprod_{i=1}^{n} U_{a_i,b_i}$$

where $\Omega = \coprod_{i=1,n} C_{a_i} = \coprod_{i=1,n} C_{b_i}$.

If for $a \in \Omega_f$, $I_a = [\overline{a}, \overline{a} + 2^{-l(a)}) \subset [0,1)$, then

$$\begin{cases} [\![G_2]\!] & \to & V \\ \coprod_{i=1}^{n} U_{a_i,b_i} & \mapsto & \begin{pmatrix} I_{a_1} & \dots & I_{a_n} \\ I_{b_1} & \dots & I_{b_n} \end{pmatrix} \end{cases}$$

is an isomorphism of groups.



$$S = U_{0,01} \coprod U_{10,00} \coprod U_{11,1} \text{ corresponds to } \begin{pmatrix} I_0 & I_{10} & I_{11} \\ I_{00} & I_{01} & I_1 \end{pmatrix}$$

Figure 3. The isomorphism $[\![G_2]\!] \cong V$

The last example realizes V as a countable subgroup of homeomorphsims of Ω. If $U = C_a$ is a cylinder for $a \in \Omega_f$, then the rigid stabilizer V_U is isomorphic to V. But V contains a nonabelian free groups, hence is nonamenable. The above theorem ensures that V is thus C^*-simple.

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