OBSTRUCTION FOR LLP OF FULL GROUP C^*-ALGEBRAS, AFTER IOANA, SPAAS & WIERSMA.

CLÉMENT DELL'AIERA

Abstract. This note surveys the talk given on December 17th, 2020 for the *Lifting for C^*-algebras seminar* ran at UMPA, ENS Lyon.

Contents

We will present theorem A of [1]. The goal is to show that, in possession of a family of finite dimensional projective representations with asymptotically trivial cocycles, LLP implies the existence of invariant vectors for carefully built representations with same cocyles. Using a projective characterization of property (T) from [3], one can deduce from relative property (T) that the cocyles are ultimately coboundaries. This provides an obstruction for LLP of the full C^*-algebra of $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$.

1. Projective characterization of property (T)

Let G be a discrete countable group. We denote by $\mathbb{P}U(n)$ the quotient of the unitary group $U(n)$ by $\mathbb{T} = U(1)$.

Definition 1.1. A projective representation is given by a group morphism $G \to \mathbb{P}U(n)$.

Any projective representation lifts to a map $\phi : G \to U(n)$ such that there exists a map $c : G \times G \to \mathbb{T}$ satisfying

$$\phi(s)\phi(t) = c(s,t)\phi(st) \quad \forall s,t \in G.$$

Writing the product of three elements in two different ways, we get that the relation

$$c(g,h)c(gh,k) = c(g,hk)c(h,k) \quad \forall g,h,k \in G$$

is satisfied. This is what is called the cocyle relation. If $c(s,t) = b(s)b(t)\overline{b(st)}$ for some map $b : G \to \mathbb{T}$, then $g \mapsto \overline{b(g)}\phi(g)$ is a unitary lift of $G \to \mathbb{P}U(n)$. Such a cocyle is called a coboundary. In a similar manner, two projective representations whose cocyle differ by a cobounday are unitarily equivalent.

Given a cocyle $c \in Z^2(G,\mathbb{T})$, one can always build a projective representation on $\ell^2(G)$ with this cocyle by defining

$$\lambda(g)\delta_s = c(g,s)\delta_{gs}.$$

We will refer to this representation as the regular c-representation.

Date: December 21, 2020.

Definition 1.2. The cohomology group $H^2(G,\mathbb{T})$ is defined a the quotient of cocycles $Z^2(G,\mathbb{T})$ by the subgroup of coboundaries $B^2(G,\mathbb{T})$. It classifies projective representations in the sense that the class of their cocyle determines their unitary equivalence class.

Recall that, if $\Lambda < G$ is a subgroup, the pair (Λ, G) is said to have (relative) property (T) if any unitary representation that has almost invariant vectors has a genuine nonzero invariant vector. We can give a characterization of property (T) by projective representations.

If $\rho : G \to U(H)$ and $\sigma \to U(K)$ are two projective representations with cocyles c_ρ and c_σ, then $\rho\otimes\sigma : G \to U(H\otimes K)$ is also a projective representation with cocycle $c_\rho c_\sigma$. The contragredient representation $H^\vee$ is a projective representation with cocyle $\overline{c}_\rho$.

In particular, the Hilbert-Schmidt operators on a projective representation

$$HS(H) = \{T \in B(H) \ : \ Tr(T^*T) < \infty\} \cong H^\vee \otimes H$$

is a unitary representation.

Lemma 1.3 (lemma 1.1 [3])**.** *The pair (Λ, G) has (T) if there exists a finite set $F \subset G$ and a positive number $\varepsilon > 0$ such that, for any projective representation $\phi : G \to U(H)$ with cocyle $c \in Z^2(G,\mathbb{T})$, for every vector such that*

$$\sup_{g\in F} d(g\cdot\xi, \mathbb{C}\xi) < \varepsilon$$

then there exist $\xi_0 \neq 0$ and $b \in Z^1(G,\mathbb{T})$ satisfying

- $\|\xi - \xi_0\| < \varepsilon$,
- $\rho(\lambda)\xi_0 = b(\lambda)\xi_0$ *and* $\partial b = c$.

In particular, the restriction $c_{|\Lambda}$ is a coboundary in $B^2(\Lambda,\mathbb{T})$.

Proof. From the almost projectively invariant vector, we get an almost invariant vector $T = \xi^\vee \otimes \xi$: there exists a Λ-invariant $T_0 \in HS(H)$ such that $\|T - T_0\|_2 \leq \varepsilon$. As $T_0^*T_0$ is Λ-invariant, all its spectral projections also are. One of these must satisfy $\|p - T\| < \varepsilon$, hence (if small enough), p is unitarily equivalent to T, a rank one projection, i.e. there exists a non zero $\xi_0 \in H$ with $p = \xi_0^\vee \otimes \xi_0$. Invariance of p give easily the 1-cocyle. □

2. Almost unitary projective representations and LLP

Let us call a family of finite dimensional projective representations with cocycles converging pointwise to 1 a *almost unitary* family. One of the key ideas of [1] is to use *almost unitary* families to build $*$-homomorphisms in QWEP C^*-algebras.

Let ω be a non-principal ultrafilter on $\mathbb{N}$, and define B to be the product C^*-algebra $\prod_n M_{d_n}$, J^ω to be the closed ideal $\{x \in B \ \ \lim_\omega \tau_n(x_n^* x_n) = 0\}$ and

$$B^\omega = B/J^\omega.$$

We will show that B^ω is QWEP, thus Kirchberg's theorem (see [2]) ensures that any ucp map from a separable LLP C^*-algebra to B^ω ucp lifts.

If $\phi_n : G \to U(d_n)$ is an almost unitary family, as

$$\|\phi_n(s)\phi_n(t) - \phi_n(st)\|_{2,\tau_n} = |c_n(s,t) - 1| \to 0,$$

the map $g \mapsto (\phi_n(g))_n \in B$ defines a multiplicative map, thus defines a $*$-homomorphism

$$C^*(G) \to B^\omega.$$

Proposition 2.1 (Theorem A [1])**.** *Let ϕ_n be an almost unitary family of finite dimensional projective representations. If $C^*(G)$ has the LLP, and there is a subgroup $\Lambda < G$ such that (Λ, G) has (T), then the restrictions of the cocycles to Λ are coboundaries:*

$$[c_{|\Lambda}] \in B^2(\Lambda, \mathbb{T}).$$

Proof. Let us show that we can build projective representations with same cocyle, that admits almost invariant vectors. Since C^*G has the LLP, the associated $*$-morphism $\phi : C^*G \to B^\omega$ lifts to a ucp map

$$\Psi : C^*G \to B.$$

Evaluating at the n^{th}-spot, we get $\lim_\omega \|\Psi_n(g) - \phi_n(g)\|_2 = 0$. Apply Stinespring theorem to lift Ψ_n to a genuine representation

$$\rho_n : C^*G \to B(\tilde{H}_n)$$

such that $\Psi_n(g) = p_n \rho_n(g) p_n$ for the projection $p_n : \tilde{H}_n \to H_n$. We consider the representation $\rho_n^\vee \otimes \phi_n$ on $\tilde{H}_n^\vee \otimes H_n$: it is a projective representation with cocyle c_n. Then

$$\frac{\|g \cdot p_n - p_n\|_{2,Tr}}{\|p_n\|_{2,Tr}} = 2(1 - Re\ \tau_n(\Psi_n(g)^* \rho_n(g))) \leq 2\|\Psi_n(g) - \phi_n(g)\|_{2,\tau_n} \to 0.$$

Thus the representation $HS(\tilde{H}_n, H_n)$, a projective representation with cocyle c_n, has almost invariant vectors. The lemma above ensures that the restricted cocyles are coboundaries by property (T). □

3. Example

On $\Lambda = \mathbb{Z}^2$, define $c(x, y) = det(x|y)$ and $c_n(x, y) = e^{i\frac{\pi}{n}c(x,y)}$. Let Γ be a non amenable subgroup of $SL(2, \mathbb{Z})$, and $G = \Lambda \rtimes \Gamma$. Extend c_n to G by

$$c_n(g, g') = c_n(x, \gamma \cdot x') \quad \forall g = (x, \gamma), g' = (x', \gamma') \in G.$$

Then these cocycles factorize through the finite subgroup of $\mathbb{Z}/n\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}/n\mathbb{Z})$, image of G under the quotient map. We can compose the regular c_n-representations with the quotient map to get an almost unitary family of finite dimensional projective representation

$$G \to U(\ell^2(G(n))).$$

On an abelian group, any coboundary is symmetric, and the c_n are antisymmetric. The cocyle restricted to Λ cannot be coboundaries and the pair has property (T) , thus $C^*(G)$ cannot have LLP.

References

[1] Adrian Ioana, Pieter Spaas, and Matthew Wiersma. Cohomological obstructions to lifting properties for full C^*-algebras of property (t) groups. *Geometric and Functional Analysis*, 30(5):1402–1438, 2020.

[2] Eberhard Kirchberg. On non-semisplit extensions, tensor products and exactness of groupc*-algebras. *Inventiones mathematicae*, 112(1):449–489, 1993.

[3] Remus Nicoara, Sorin Popa, and Roman Sasyk. On ii1 factors arising from 2-cocycles of w-rigid groups. *Journal of Functional Analysis*, 242(1):230–246, 2007.

Department of Mathematics, UMPA, ENS Lyon 46 allée d'Italie 69342 Lyon Cedex 07 FRANCE

Email address: clement.dellaiera@ens-lyon.fr