METRIC GEOMETRY AND GEOMETRIC ANALYSIS (OXFORD, UNITED KINGDOM)

CLÉMENT DELL'AIERA

ABSTRACT. This document compiles some questions and problems that were discussed during the TA sessions.

CONTENTS

1. AMENABILITY

Investigating the Banach-Tarski paradox, Von Neumann introduced the concept of amenability for groups. His explanation of the paradoxical decomposition of the sphere relied on the existence of two elements in $SO(3)$ generating a non abelian free group. Non abelian free groups being non amenable, one could produce a paradoxical decomposition of $\mathbb{F}_2$ and pull it back to the sphere.

A countable discrete group Γ is amenable if there exists an invariant mean, i.e. a positive linear functional $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ that satisfies

$$\varphi(\gamma \cdot f) = \varphi(f) \quad \forall f \in \ell^\infty(\Gamma) \forall \gamma \in \Gamma,$$

where $(\gamma \cdot f)(s) = f(\gamma^{-1}s)$.

(1) Show that finite groups are amenable.
(2) Show that abelian groups are amenable.
(3) Show that amenability is stable by extension.
(4) Show that solvable groups are amenable.
(5) Show that $\mathbb{F}_2$ is not amenable, nor any group containing it.

Maybe write something about paradoxical decomposition.

1.1. Tits alternative.

Tits alternative. For a while, the only examples of non amenable groups contained a non abelian free group. This lead to the Day-Von Neumann problem: is amenability equivalent to containing $\mathbb{F}_2$?

While this was solved in the negative[1], Tits proved that for finitely generated linear groups, this was essentially true.

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[1]The first solution to the Day-Von Neumann problem was given by Ol'shanskii, who proved that the Tarski monster he constructed were not amenable, see [2], [3], [4]

Theorem 1.1 (Jacques Tits, 1972 [5])**.** *Let $\mathbb{K}$ be a field and Γ be a subgroup of $GL(n,\mathbb{K})$. We suppose that either Γ is finitely generated or $\mathbb{K}$ is of characteristic zero. Then the following alternative holds:*

- *Γ is virtually solvable,*
- *Γ contains a non abelian free group.*

Notice that both assumptions cannot be dropped at the same time: $SL(n,\overline{\mathbb{F}}_q)$ does not satisfy the previous theorem since it is almost simple and a torsion group.

1.2. **Problems and exercises.**

(1) Find a paradoxical decomposition for $\mathbb{F}_2$.
(2) Show that Tits alternative is really an alternative. (Hint: why is $\mathbb{F}_2$ not solvable? Why is a subgroup of a free subgroup free?)
(3) Prove the tennis-table lemma: Suppose that a group G acts on a set S. Let Γ_1, Γ_2 be two subgroups of a group G, and Γ be the subgroup generated by Γ_1 and Γ_2. Suppose there are disjoint subsets $A, B \subset S$ such that every non trivial element of Γ_1 sends A into B, and every non trivial element of Γ_2 sends B into A. Then $\Gamma = \Gamma_1 * \Gamma_2$.
(4) Show that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a free group $\mathbb{F}_2$.
(5) Let $u, v \in \mathbb{R}^3$ two unit vector of angle $\frac{\pi}{4}$, and $g = Rot(u,\pi)$ and $h = Rot(v, \frac{\pi}{3})$. Show that $\langle g, h\rangle = \mathbb{Z}/2 * \mathbb{Z}/3$.
(6) Let[2] $\Gamma < PSL(2,\mathbb{R})$.
    (a) Show that if Γ contains a parabolic element g and another element whose fixed points are disjoints from that of g, then Γ contains a non abelian free group.
    (b) Show that if Γ contains two hyperbolic elements without common fixed points, then Γ contains a non abelian free group.
    (c) Prove Tits' alternative for subgroups of $PSL(2,\mathbb{R})$.

1.3. **Monod's examples.** Monod gave simple examples (see [1]) of non amenable groups without any non abelian free subgroups. It is quite interesting that these are built using the action of $PSL(2,\mathbb{R})$ on the projective line. Tits alternative would lead us to think that this would not work. The idea of Monod is to use elements that act piecewise as $PSL(2,\mathbb{R})$. They retain enough information to be non amenable, yet are still bounded to have relations between each others.

We propose a simple case of Monod's construction.

(1) Show that $PSL(2,\mathbb{Z}[\sqrt{2}])$ is a lattice in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.
(2) Show that $PSL(2,\mathbb{Z}[\frac{1}{p}])$ is a lattice in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{Q}_p)$.
(3) Let H be a closed subgroup in a locally compact group G. We suppose that G acts on a compact Hausdorff space X by homeomophisms.
    (a) Show that if H is co-amenable in G and the restricted action of H on G is amenable, then the action of G on X is amenable.
    (b) Show that if Y is a closed H-invariant subset in X, and the action of G on X is amenable, then so is the action of H on Y. In particular, the stabilizers of an amenable action are amenable.
(4) Show that $PSL(2,\mathbb{Z})$ does not act amenably on $\mathbb{P}^1(\mathbb{R})$.

This problem is not finished yet

[2]For this question, one needs the classification of isometries of hyperbolic spaces, which are classified by their number of fixed points in $\overline{\mathbb{H}}$: 1 in $\mathbb{H}$ (elliptic), 1 in $\mathbb{S}^1$ (parabolic), 2 in $\mathbb{S}^1$ (hyperbolic).

2. Coarse equivalence vs quasi-isometries

A coarse embedding is a map $f : X \to Y$ such that there are proper increasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$, diverging to $+\infty$, such that

$$\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y)) \quad \forall x, y \in X.$$

Show that a tree is coarsely embeddable into a (the) real separable Hilbert space (i.e. ℓ^2).

3. Distorted subgroups

(1) Show that asymptotic dimension is increasing: if $H < G$ then $asdim(H) \leq asdim(G)$.
(2) There are subgroups which are distorted, i.e. such that $(H, |\cdot|_H)$ and $(H, |\cdot|_G)$ are not quasi-isometric. For instance, show that is the case when G is the (Baumslag-Solitar) group generated by $s = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and H is the subgroup generated by t.
(3) (Horoballs are distorted in $\mathbb{H}^2$) Another example: let P be generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $G = SL(2, \mathbb{Z})$. Is H distorted in G?

4. Covering vs asymptotic dimension

(1) Show that $\mathbb{Z} \wr \mathbb{Z}$ has infinite asymptotic dimension.
(2) Show that for all $n \in \mathbb{N} \cup \{\infty\}$, there exists a topological surface (of covering dimension 2) of asymptotic dimension n.

5. Asymptotic dimension of Lie groups

(1) Show that solvable groups have finite asymptotic dimension.
(2) Show that $GL(n, \mathbb{R})$ has finite asymptotic dimension. (Hint: polar decomposition)
(3) Show that an almost connected Lie group has finite asymptotic dimension. (Hint: use the adjoint representation)

Also: show that semisimple Lie groups are quasi-isometric to an amenable group.

References

[1] Nicolas Monod. Groups of piecewise projective homeomorphisms. *Proceedings of the National Academy of Sciences*, 110(12):4524–4527, 2013.
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[5] Jacques Tits. Free subgroups in linear groups. *Journal of Algebra*, 20(2):250–270, 1972.

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